NPS-61MD72121A

NAVAL POSTGRADUATE SCHOOL

Monterey, California

31 December 1972 NPS-61Md72121A

Coherent and Incoherent Components of Sound Scattered at a Time Dependent Rough Surface

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ABSTRACT:

Theoretical expressions are derived for the sound scattered at a time-dependent rough surface. The calculations are made for a Gaussian shaded source transducer and point receiver. The Helmholtz theorem and Fresnel approximation are used. The rough surface is assumed to be a traveling wave and to have a traveling wave packet type of correlation function. The coherent component of the signal is the product of the Fourier transformation of the surface distribution function and the smooth surface reflection signal. Comparison of theory and experiment shows the coherent component to be sensitive to the non-Gaussian character of the wind-blown water waves. The incoherent components and the temporal correlation function of the scattered sound are given. For the special case of a traveling cosine wave type of rough surface, the spectrum of the scattered sound includes components which are multiples of the frequency of the surface wave. For surfaces describable by a bivariate Gaussian distribution function, the temporal correlation is a function of, but not the same as, the time dependence of the rough surface. The scattered sound is insensitive to the spatial correlation function of the surface at distances larger than the dimensions of the transducer divided by the cosine of the incident angle. The final expressions are complex error integrals and can be used for all values of roughness. This task was supported by Naval Ship Systems Command (Code PMS 388).

TABLE OF CONTENTS

I. EXPERIMENT AND THE INVERSE PROBLEM------------4

II. SCATTERING FUNCTION FROM THE HELMHOLTZ EQUATION------------6

III. EVALUATION OF THE SCATTERING INTEGRAL------------18

IV. COHERENTLY SCATTERED SIGNAL------------24

V. COSINE CORRUGATED SURFACE------------28

VI. TOTAL SIGNAL SCATTERED AT A NON-GAUSSIAN SURFACE------------31

REFERENCES------------33

INITIAL DISTRIBUTION LIST------------34

FORM DD 1473------------38

I. EXPERIMENT AND THE INVERSE PROBLEM

Sound scattered at a time dependent surface takes on some of the time dependence of the surface. For example, the upward or downward motion of the surface causes the frequency of the reflected signal to be Doppler shifted. It is easy to demonstrate the effects in laboratory experiments (Fig. 1.1). Small waves on the water cause the phases of the signals to fluctuate. Experiments have shown the reflected signals are (crudely stated) modulated by the surface and often the spectrum of the surface can be identified as a component of the spectrum of the reflected signal. With our growing technology in remote sensing, the importance of being able to go from scattering measurements back to a description of the time dependent surface is extremely important. It may be more important to know when and how far one can go for a given technique.

Doing the inverse problem requires a complete knowledge of the relationship of the measurement to the quantity being sensed. Too often, the result of having this knowledge is "I can't measure what I want by doing the experiment I am doing." For example, in the plane wave approximation, Eckart (1953) showed that mean square scattered signals are simply related to the correlation function of the surface at small $\gamma\sigma$ ($\gamma = k(\cos\theta_1 + \cos\theta_2)/2$, k is 2π/wavelength, σ is rms roughness, and θ_1 and θ_2 are the incident and reflected angles). Unfortunately the scattered signal is proportional to $\gamma^2\sigma^2$ and correspondingly very small. If the source and receiver are omnidirectional or broad beamed, the scattered sound is usually identified as being a fluctuation of the signal and is often inseparable from the noise.

In this paper I add another limitation (which is derived in the Fresnel approximation. The scattered sound is insensitive to spatial correlation function of the rough surface at correlation distances larger than the dimensions of the transducer/$\cos\theta_1$.

I believe a bit of discussion of the theoretical problem and approximations is needed. In seeking the source of a discrepancy between my theory (1971) and some of our laboratory measurements, G. A. Sandness identified the calculation of the incident sound at the surface as being the difficulty for two reasons, (private communication). First, in using the Helmholtz equation, the incident sound signal should satisfy the wave equation. The combination of a point source and directional function (or illumination function on the surface) does not satisfy the wave equation. Second, the limitation of the expansion to second order terms may be a very poor approximation for a diverging wave at the surface. Another way of thinking about this is to regard the phase and amplitude of the incident signal as being the hologram of the source. The hologram of a finite object is different from that of a point source. The reader may wish to read Melton and Horton (1970).

Having said that there are difficulties with the Fresnel approximation, I have chosen to use it. Also, I use a Gaussian shaded source transducer and a point receiver. This shading fits the measured response of our transducers and facilitates repeated integrations. I have obtained expressions that can be used over a wide range of surface roughnesses and correlation functions without resorting to separate expansions for the high and low frequency limits. The end result is the mean square scattered sound and its temporal correlation function.

II. SCATTERING FUNCTION FROM THE HELMHOLTZ EQUATION

The development of a theoretical relationship of a scattering function to the properties of the surface requires analysis of the scattering of acoustic signals by rough surfaces. This has been the subject of a number of papers and is treated in several books, notably Beckmann and Spizzichino (1963), Tolstoy and Clay (1966), Ol'Shevskii (1967), Fortuin (1970), and Horton (1971). Although we will not discuss it here, in the electronic and radio engineering journals an extensive literature exists on the scattering of electromagnetic waves by various types of irregular and rough surfaces. We are obliged to present the theory in some detail because the relevant underwater acoustics theory has not been applied to these types of problems. Most of the studies have been empirical.

The derivation of the scattering equation from the Helmholtz theorem is usually based upon the assumption of local plane reflections. It is suggested that the reader who is interested in the details refer to the development given by Tolstoy and Clay (1966), who base their derivations on those of Eckart (1953) and Beckmann and Spizzichino (1963). The sound absorption will be ignored in the derivation.

The general assumptions are: 1) The source and receiver are far from the illuminated area. 2) The dimensions of the source are small compared to R_1 and R_2. 3) The source is a Gaussian shaded transducer and is directed along R_1, (Fig. 2.1). Also, the individual elements are in phase and incident sound pressure at (x,y,ζ) is the integral over ds'. 4) No shadows are present.

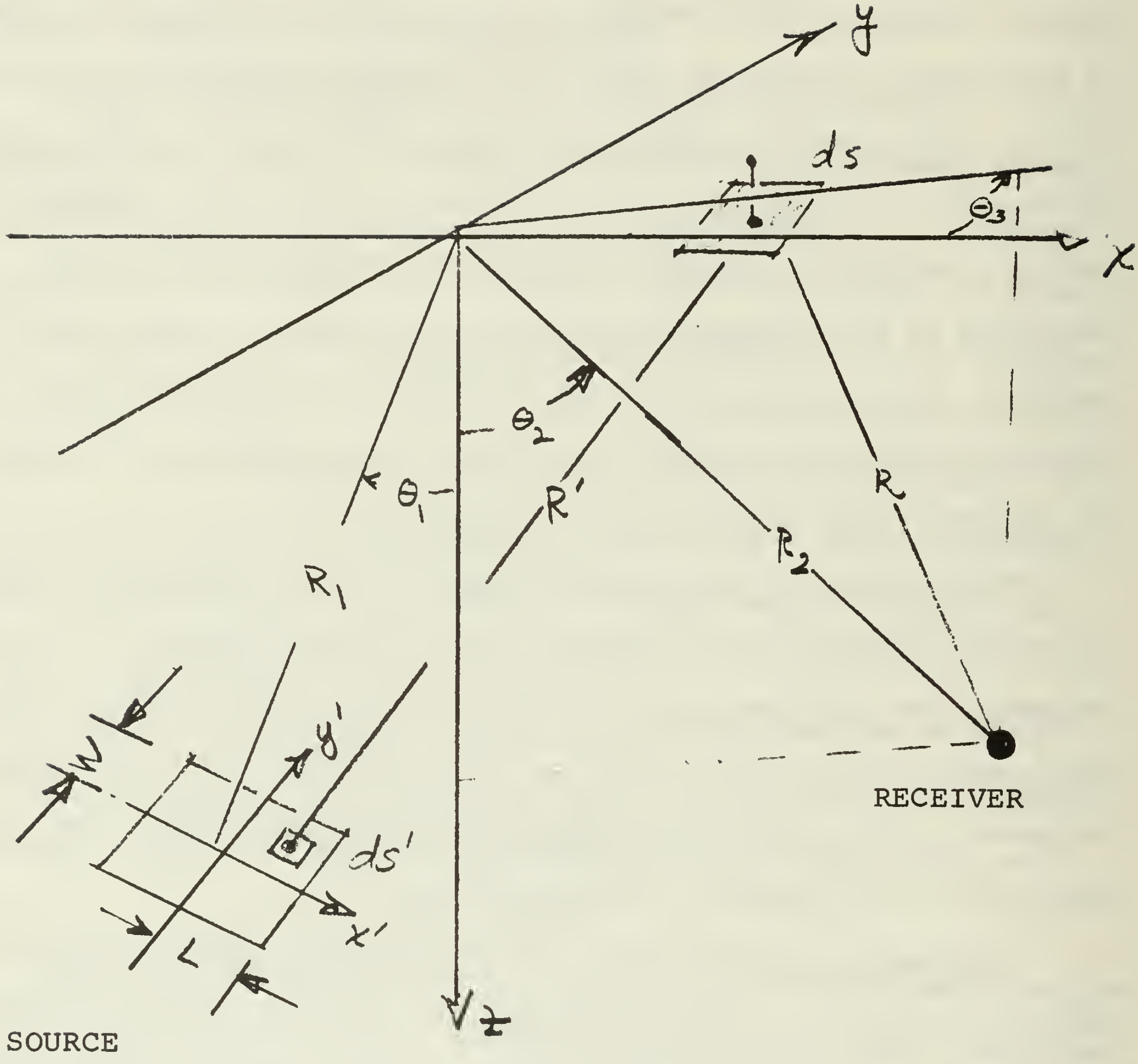

Fig. 2.1 Geometry

ds is at the position x, y, ζ relative to the origin. The plane of the source is perpendicular to R_1.

Because the surface is rough, we can expect that at grazing angles some of the surface will be in shadow. The proportion of the area in shadow to the total area scanned depends upon the shape of the surface and on the grazing angle. Wagner (1967) discussed this for rays and a randomly rough surfaces. In any case, the assumption of no shadowing is violated. It is also likely that the radii of curvature of some features on the surface are small compared to the acoustical wavelength. In that case, the sound does not have a local plane reflection and in order to describe the scattered sound we must use higher order expansions.

Eckart (1953) remarked that the boundary conditions are troublesome. For example, one can set*

$$p = \mathcal{R} p_1 \tag{2.1}$$

$$\frac{\partial p}{\partial n} = - \mathcal{R} \frac{\partial p_1}{\partial n} \tag{2.2}$$

where p_1 and p are the incident and reflected signals on the local surface and $\partial/\partial n$ is the normal derivative. Using $\mathcal{R} = -1$ for a free surface, Eckart suggested that in the smoother areas, the second condition might hold. In deep shadows, he thought that

$$\frac{\partial p}{\partial n} = \mathcal{R} \frac{\partial p_1}{\partial n} \tag{2.3}$$

might be reasonable. Horton and Muir (1967) assumed an average boundary condition and combined the two equations involving the normal derivatives to obtain

---

* $\mathcal{R} = \dfrac{\rho_2 c_2 \cos\theta_1 - \rho_1 c_1 \cos\theta_r}{\rho_2 c_2 \cos\theta_1 + \rho_1 c_1 \cos\theta_r}$, reflection coefficient for plane waves,

here 1 and 2 refer to medium 1 and 2. θ_r is refracted direction in medium 2.

$$\partial p_1/\partial n = 0 \text{ or } \partial p/\partial n = 0 \tag{2.4}$$

They also used Eckart's small slope approximation and replaced the normal derivative by $\partial/\partial z$. Horton et al. (1967) compared theoretical computations and the experimental scattered sound and found the agreement to be quite good. The slope correction can be included by performing an integration by parts [Tolstoy and Clay, p. 196-199 (1966)]. In the specular direction, the result is the same as obtained with $\partial/\partial n \simeq \partial/\partial z$. The result is different for backscattered sound.

The most direct way to demonstrate the approximations is to start with the integral expression for the scattered acoustical pressure. With the aid of the Helmholtz theorem [Born and Wolf, p. 375 (1965)] it is

$$p = \frac{1}{4\pi} \int_S \left(p\Big|_S \frac{\partial U}{\partial n} - U \frac{\partial p}{\partial n} \Big|_S \right) ds \tag{2.5}$$

where

$$U = e^{i(kR - \omega t_2)}/R \tag{2.6}$$

$$p_1 = B \quad e^{i(kR' - \omega t_1)}/R' \tag{2.7}$$

$$B^2 \equiv \Pi\rho c\ (2\pi)^{-1}$$

$$\Pi \equiv \text{source power}$$

and the normal is drawn toward the receiver. R' and R are the source and receiver distances to ds (at x, y, z). In applying the Helmholtz theorem, the surface is closed by a hemisphere at infinity. There are no other sources and all waves are outgoing from our source. Hence the contribution to the integral for the surface at infinity can be ignored and the integral over the illuminated interface is used. We choose to use the less conventional form of the normal toward the receiver because that is the direction of the wave propagation and it does not matter where the source is placed. The substitution of (2.1) and (2.2) into (2.5) gives

$$p = \frac{1}{4\pi}\int_S \mathcal{R}\ \frac{\partial}{\partial n}\ (p_1 U)\ \Big|_S\ ds \tag{2.8}$$

or the Horton-Muir condition, 2.4 gives

$$p = \frac{1}{4\pi}\int_S \mathcal{R}\ p_1\ \frac{\partial U}{\partial n}\Big|_S\ ds \tag{2.9}$$

After expansion of the integrands for a moderately directive source, the integrals will be the same and the kind of boundary condition depends upon a constant factor in front of the integral.

There are several ways of writing the algebra for the expansion and they are all messy. Since the transducer plane is perpendicular to R_1, R'^2 and R^2 are

$$R'^2 = (-R_1 \sin\theta_1 + x' \cos\theta_1 - x)^2 + (y' - y)^2 + (R_1\cos\theta_1 - x'\sin\theta_1 - \zeta)^2$$

$$R^2 = (R_2\sin\theta_2\cos\theta_3 - x)^2 + (R_2\sin\theta_2\sin\theta_3 - y)^2 + (R_2\cos\theta_2 - \zeta)^2 \tag{2.10}$$

With the aid of the binomial expansion and retaining ζ and second order terms, R' and R are

$$R' \simeq R_1 + \frac{x'^2 + y'^2}{2R_1} + \frac{x^2\cos^2\theta_1 + y^2}{2R_1} - \frac{x' x\cos\theta_1}{R_1} - \frac{y' y}{R_1}$$

$$+ x\sin\theta_1 - \zeta\cos\theta_1 + \ldots$$

$$R \simeq R_2 + \frac{x^2}{2R_2}\left(1 - \sin^2\theta_2\cos^2\theta_3\right) + \frac{y^2}{2R_2}\left(1 - \sin^2\theta_2\sin^2\theta_3\right)$$

$$- x\sin\theta_2\cos\theta_3 - y\sin\theta_2\sin\theta_3 - \zeta\cos\theta_2 + \ldots \tag{2.11}$$

For small slopes, the normal derivative is approximately $\partial/\partial Z$ (or $\partial/\partial\zeta$ here). On making this approximation and also including the integral over the transducer ds', one obtains

$$p \simeq - \frac{ik\ B\mathcal{R}F\ e^{-i\omega t}}{2\pi\ R_1\ R_2} \int g\ ds' \int e^{ik\ (R' + R)} ds \tag{2.12}$$

$$F = \begin{cases} (\cos\theta_1 + \cos\theta_2)/2, \text{ boundary cond. (2.1 and 2), } \partial/\partial n \simeq \partial/\partial z \\[2ex] \dfrac{1 + \cos\theta_1 + \cos\theta_2 - \sin\theta_1 \sin\theta_2 \cos\theta_3}{\cos\theta_1 + \cos\theta_2}, \text{ slope correction (Tolstoy and Clay, 1966)} \\[2ex] \cos\theta_2/2, \text{ boundary cond. (2.4)} \end{cases} \tag{2.13}$$

$$g = \frac{1}{\pi LW} \exp\left\{ - \frac{x'^2}{L^2} - \frac{y'^2}{W^2} \right\} \tag{2.14}$$

Integration over the source yields (after algebraic manipulation)

$$p \simeq - \frac{ik\ B\mathcal{R}F\ e^{\left[-i\omega t\ -ik(R_1 + R_2)\right]}}{2\pi\ R_1 R_2 (1 - id_L)^{\frac{1}{2}} (1 - id_W)^{\frac{1}{2}}}$$

$$\iint \exp\left[- a_x x^2 - a_y y^2 + 2\ i\alpha x + 2\ i\beta y + 2\ i\gamma\zeta\right] dydx \tag{2.15}$$

$$a_x \equiv - \frac{ik \cos^2\theta_1 R_x}{2\ R_1 R_2} \left[\frac{1 - i\ (1 - R_2/R_x)\ d_L}{1 - id_L}\right]$$

$$a_y \equiv - \frac{ik\ R_y}{2\ R_1 R_2} \left[\frac{1 - i\ (1 - R_2/R_y) d_W}{1 - id_W}\right]$$

$$R_x \equiv R_2 + R_1\ (\sin^2\theta_2 \cos^2\theta_3)/\cos^2\theta_1$$

$$R_y \equiv R_2 + R_1\ (1 - \sin^2\theta_2 \sin^2\theta_3)$$

$$d_L \equiv kL^2/(2R_1) \qquad d_W \equiv kW^2/(2R_1) \tag{2.16}$$

$$2\alpha = k(\sin\theta_1 - \sin\theta_2 \cos\theta_3)$$

$$2\beta = -k(\sin\theta_2 \sin\theta_3)$$

$$2\gamma = -k(\cos\theta_1 + \cos\theta_2)$$

Integration of 2.15 for $\zeta = 0$ yields the image solution. Although the size of the transducer is included and the expressions are dependent on L and W, I don't think the expressions are accurate for near field computations because the higher powers of x and y were dropped in 2.11.

Proceeding, the covariance of the sound pressure is

$$\left\langle p\ (t + T)\ p^*\ (t)\right\rangle = \frac{k^2 B^2 \mathcal{R}^2 F^2 e^{-i\omega\tau}}{4\pi^2 R_1^2 R_2^2\ (1 + d_L^2)^{\frac{1}{2}}(1 + d_W^2)^{\frac{1}{2}}}$$

$$\left\langle \iiiint \exp\left\{ -a_x x^2 - a_x^* x'^2 - a_y y^2 - a_y^* y'^2 + 2i\alpha(x-x') + 2i\beta(y-y') + 2i\gamma(\zeta - \zeta') \right\} dy\ dx\ dy'\ dx' \right\rangle \qquad (2.17)$$

I assume the surface is random and ζ has a zero mean. The correlation function of ζ is*

$$\sigma^2\psi = \lim_{X,Y,T \to \infty} \frac{1}{XYT} \int_{-\frac{X}{2}}^{\frac{X}{2}} \int_{-\frac{Y}{2}}^{\frac{Y}{2}} \int_{-\frac{T}{2}}^{\frac{T}{2}} \zeta\ \zeta'\ dx\ dy\ dt \qquad (2.18)$$

where in functional notation ψ could be written as $\psi(\xi, \eta, \tau,)$.

Assuming a Gaussian surface, the bivariate Gaussian probability density is

$$W = (2\pi\sigma^2)^{-1}\ (1-\psi^2)^{-\frac{1}{2}}$$

$$\exp\left\{ -\left[2\ (1-\psi^2)\sigma^2 \right]^{-1} \left[\zeta^2 + \zeta'^2 - 2\zeta\zeta'\psi \right] \right\} \qquad (2.19)$$

Since the only random quantities are ζ and ζ', the average in (2.17) operates on them as follows:

---

*For a given roughness wave length Λ, the shorter wave length roughness in an area having the dimensions of several Λ has the same statistical properties as all other similar size areas.

$$\left\langle e^{2i\gamma(\zeta-\zeta')}\right\rangle = \iint W\, e^{2i\gamma(\zeta-\zeta')} d\zeta d\zeta' = C_2$$

$$\left\langle \qquad \right\rangle = \exp\left[-4\gamma^2\sigma^2(1-\psi)\right] \qquad (2.20)^*$$

The following changes of variable permit integration over x" and y".

$$x = x'' + \xi/2 \qquad y = y'' + \eta/2$$
$$x' = x'' - \xi/2 \qquad y' = y'' - \eta/2 \qquad (2.21)$$

The substitution of 2.20 and 2.21 into 2.17 and evaluation of the integral yields, after the usual algebra,

$$\left\langle p(t+\tau)\, p^*(t)\right\rangle \simeq \frac{B^2F^2\mathcal{R}^2 e^{-i\omega\tau}}{2\pi R_2^2\, LW \cos\theta_1}$$

$$\iint_{-\infty}^{\infty} \exp\left[-a_\xi \xi^2 - a_\eta \eta^2 + 2i\alpha\xi + 2i\beta\eta - 4\gamma^2\sigma^2(1-\psi)\right] d\xi\, d\eta \qquad (2.22)$$

where

$$a_\xi = \frac{R_x^2 \cos^2\theta_1 \left[1 + d_L^2(1-R_2/R_x)^2\right]}{2\, R_2^2\, L^2}$$

$$a_\eta = \frac{R_y^2\left[1 + d^2(1-R_2/R_y)^2\right]}{2\, R_2^2\, W^2} \qquad (2.23)$$

If ψ can be expressed as a sum of first and second order polynomials in ξ, η, and τ, (2.22) can be integrated directly for all values of $\gamma\sigma$.

The intensity of scattered sound is often expressed with the aid of a scattering function S as follows:

$$\left\langle p^2\right\rangle = \frac{B^2}{R_1^2}\,\frac{A}{R_2^2}\, S \qquad (2.24)$$

*In Section 6, I replace $\exp\left[-4\gamma^2\sigma^2(1-\psi)\right]$ by the characteristic function C_2 and remove the restriction that the distribution be Gaussian.

where A = illuminated area

To change (2.22) into this form I need to express A in terms of the transducer dimensions and R_1. Since transducers are often described by their Fraunhofer "beam width", I do so here. For plane waves, defining $\emptyset$ and χ as shown on Figure 2.2, and ignoring time dependence,

$$|p| = \frac{1}{\pi LW} \int_{-\infty}^{\infty} \exp\left[- \frac{x'^2}{L^2} - \frac{y'^2}{W^2} - ikx'\sin\emptyset - iky'\sin\chi \right] dx'\, dy' \quad (2.25)$$

$$|p| = \exp\left[- k^2 L^2 \sin^2\emptyset/4 - k^2 W^2 \sin^2 \chi/4 \right]$$

The response is shown on Figure 2.3. I choose to use $|p| = e^{-1}$ to define $\Delta\emptyset$ and $\Delta\chi$, as follows

$$\text{at } \chi = 0, \ (kL\sin\emptyset)/2 = 1$$

and

$$\text{at } \emptyset = 0, \ (kW\sin\chi)/2 = 1$$

Thus

$$\sin\emptyset \simeq \Delta\emptyset = 2/(kL)$$

$$\sin\chi \simeq \Delta\chi = 2/(kW) \quad (2.26)$$

The substitution of (2.26) into (2.22) and casting into the form (2.24) yields

$$\left\langle p\,(t+\tau)\, p^*(t) \right\rangle = \frac{B^2}{R_2^2} \quad \frac{A}{R_1^2} \quad S \quad (2.27)$$

$$S \equiv \frac{k^2 F^2 \mathcal{R}^2 e^{-i\omega\tau}}{8\pi} \iint_{-\infty}^{\infty} \exp\left[-a_\xi \xi^2 - a_\eta \eta^2 + 2\, i\alpha\zeta + 2\, i\beta\eta - 4\gamma^2 \sigma^2 (1-\psi) \right] d\zeta\, d\eta \quad (2.28)$$

$$A \equiv R_1^2 \ \Delta\emptyset\, \Delta\chi / (\cos\theta_1) \quad (2.29)$$

Eq. (2.22) is equivalent to Eq. (2.27) and (2.28). We can proceed in several ways, direct evaluation at 2.28, by numerical integration, polynomial expression of ψ and integration, and special functions for ψ and integration.

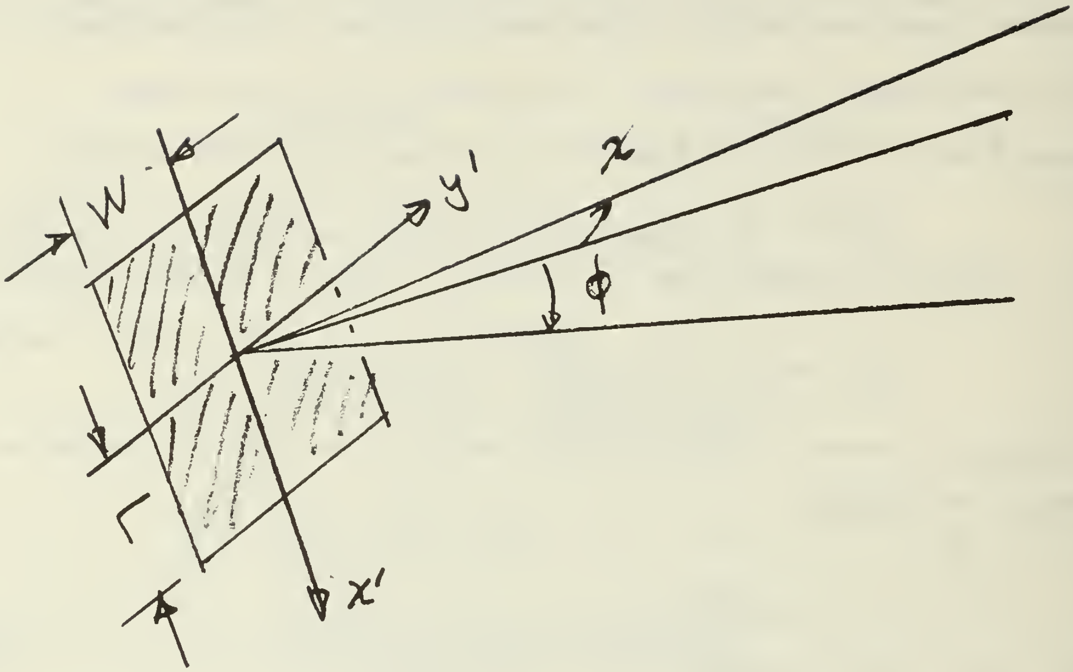

Fig. 2.2 Transducer. The transducer is Gaussian shaded. L and W indicate the dimensions for e^{-1} amplitude shading.

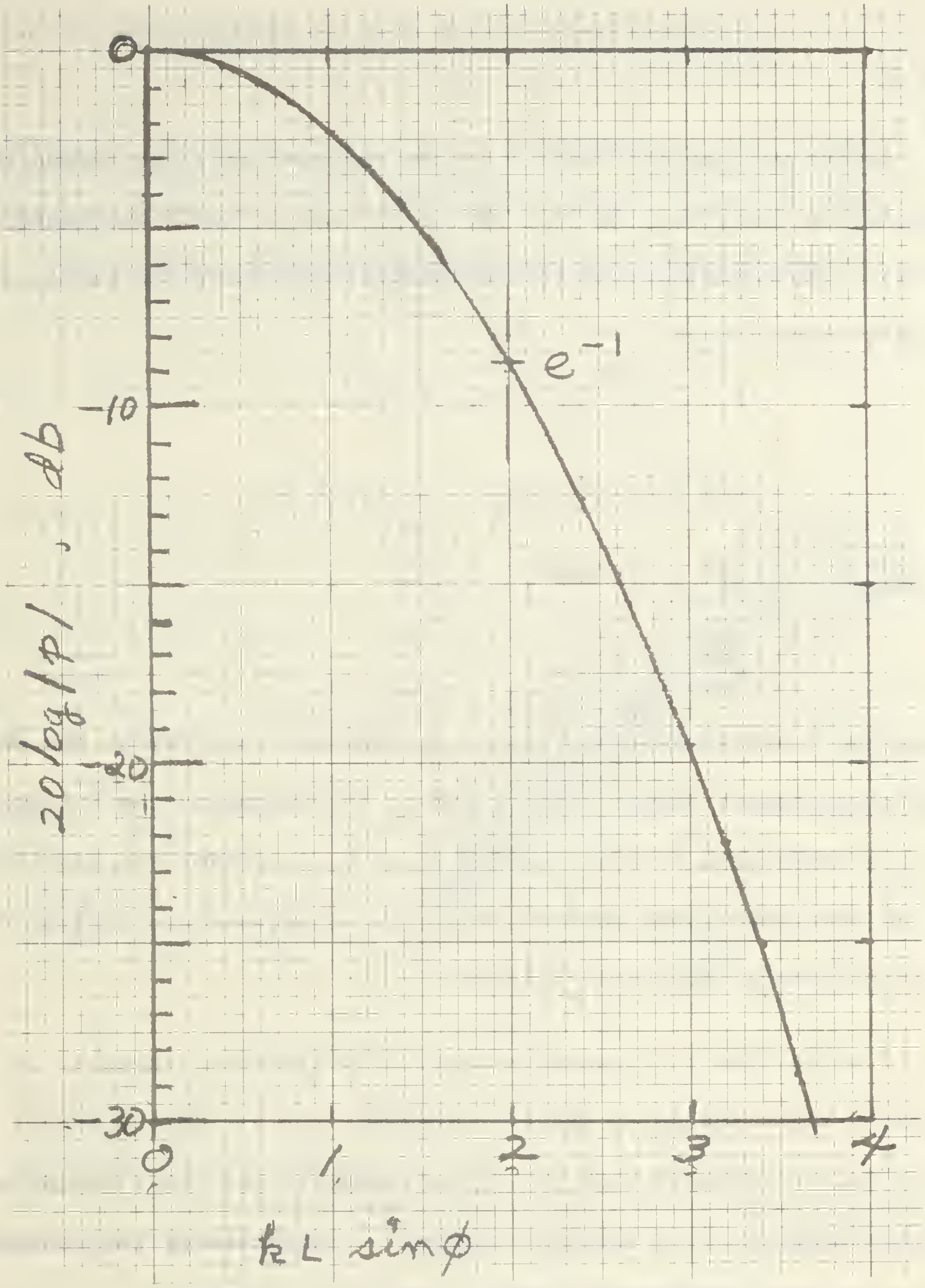

Fig. 2.3 Response of the Gaussian shaded transducer. The far field (Fraunhofer) response of the Gaussian shaded transducer, $g = \exp\left[-x'^2/L^2\right]$

Before we cast off, sail into the confused sea, and expand the correlation functions, it will pay us to look at the convergence of (2.22). The contributions to the integral are small for values of ξ and η larger than

$$\xi_f > a_\xi^{-\frac{1}{2}} \qquad \eta_f > a_\eta^{-\frac{1}{2}} \tag{2.30}$$

and for $R_1 = R_2 \qquad \theta_1 = \theta_2$

where $a_\xi^{-\frac{1}{2}} \simeq L/\cos\theta_1$

$a_\eta^{-\frac{1}{2}} \simeq W$

Since the contributions are small, we need not worry about the shape of ψ at distances larger than ξ_f and η_f in evaluating the integral. Or, the dimensions of the transducer and θ_1, determine the sensitivity of the scattered sound to ψ. On the other hand, accurate fits at small ξ and η are extremely important at large $\gamma\sigma$.

I would like to close this section by giving my thoughts on the physical significance of the ξ, η integral for S. This integral ought not be confused with the first integral over the illuminated area because the ξ, η integral relates the phase of the scattered signal at any point on the surface relative to a nearby point at a displacement ξ and η. The contributions to S are small for ξ and η greater than ξ_f and η_f. Paraphased, the integral is in correlation space. For a given roughness, the constancy of the phases of the scattered components of signal depends upon the dimensions of the transducer.

III. EVALUATION OF THE SCATTERING INTEGRAL

Experimental measurements of the ξ, τ dependence of wind blown water waves have shown that ψ has the form of traveling wave packet, Fig. 3.1. The envelope moves at the group velocity and the phases

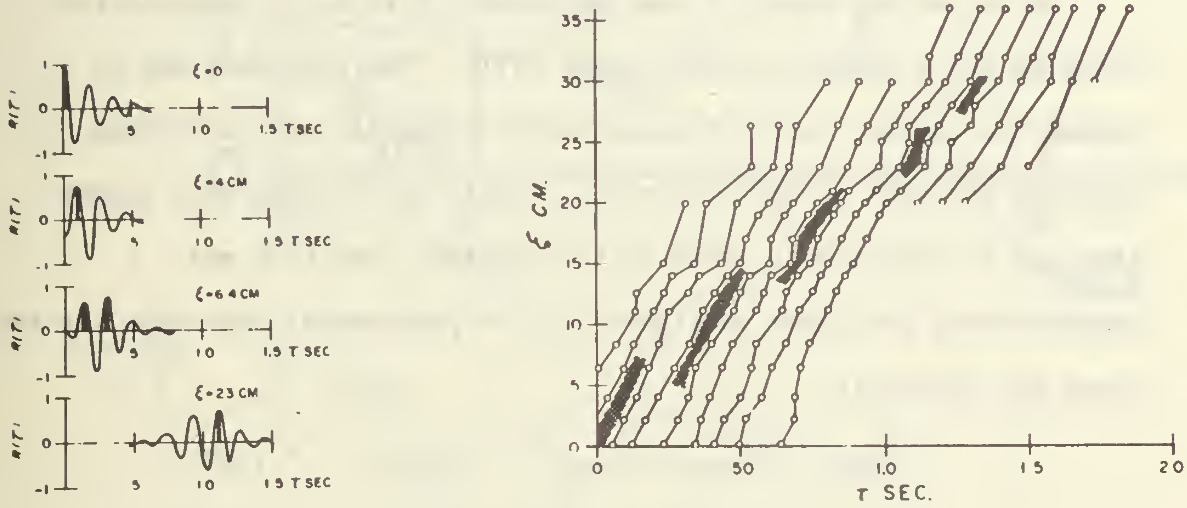

Fig. 3.1 ψ (ξ, τ)

On the graphs $R(\tau)$ is our ψ

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of the damped oscillation move at the phase velocity. Near the origin ($\xi > \xi_f$, etc), the dependence on ξ and τ can be approximated as

$$\psi \simeq \psi(\xi - v\tau) \tag{3.1}$$

for waves traveling in the + x direction. ψ is symetric about $\xi = 0$, and $\tau = 0$

$$\psi(\xi) = \psi(-\xi)$$

$$\psi(\tau) = \psi(-\tau) \tag{3.2}$$

Near $\xi = 0$, ψ can be approximated as a polynomial

$$\psi \simeq 1 - a x^2 - b | x |$$

$$\text{or} \quad \psi \simeq 1 - a\xi^2 + 2 a v \xi \tau - b | \xi - v\tau | - a^2 v^2 \tau^2 \tag{3.3}$$

and the expansion has cross terms $\xi\tau$. Correspondingly, waves moving in the y direction have $\eta\tau$ terms and waves moving in an arbitrary direction have both. This doesn't make the analysis more difficult because τ is a parameter.

In an earlier paper, I used polynomial fits to the correlation function for a random surface (Clay, 1971). The procedure was to divide the surface into sub areas and to integrate the scattering function for each of the sub areas. I will do the same here except that the τ dimension is added to the problem. The ξ, η and τ nomenclatures are shown on Figure 3.2. A polynominal expansion for the ijkth sub region is

$$\begin{aligned} \psi_{ijk} &= c'_{ijk} - a'_{ijk}\xi^2 - b'_{ijk}\xi - e'_{ijk}\eta^2 \\ &\quad - f'_{ijk}\eta - g'_{ijk}\tau^2 - h'_{ijk}\tau - m'_{ijk}\xi\tau \\ &\quad - \eta'_{ijk}\eta\tau - r'_{ijk}\xi^2\tau^2 - s_{ijk}\eta^2\tau^2 \end{aligned} \tag{3.5}$$

It isn't necessary to expand the τ dependence as a polynomial in some problems. For example, the correlation function can be expanded as the product of polynomials in ξ and functions of τ such as $\cos p\tau$ and $\sin p\tau$. If I had the polynomial form programed, I wouldn't bother.

The coefficients of the variables ξ and η can be combined

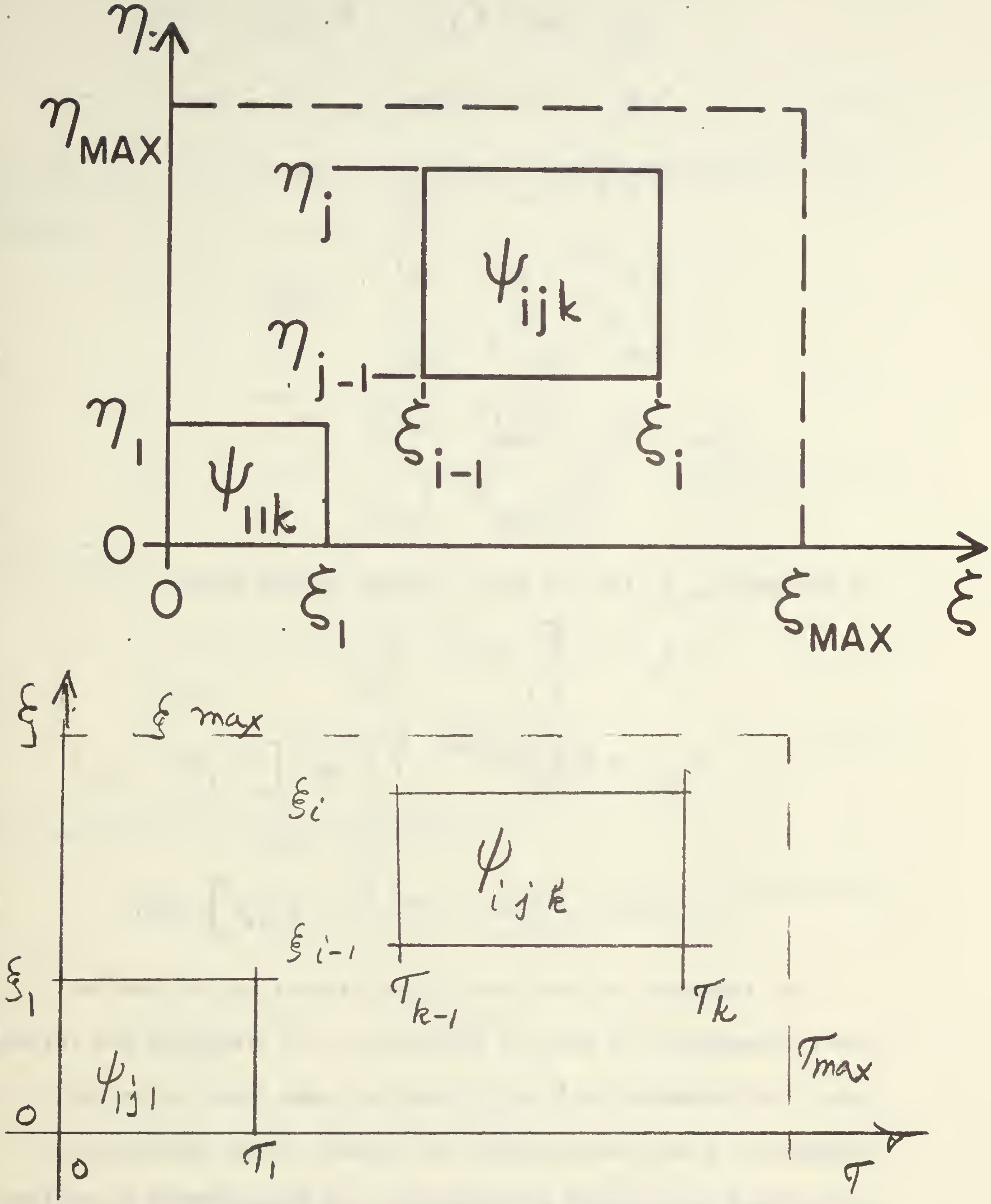

Fig. 3.2 sub areas

The ξ, η map is at constant τ. The ξ, τ map is at constant η.

ψ is assumed to be 0 for regions beyond ξ_{max}, η_{max} and τ_{max}

$$c_{ijk} \equiv c'_{ijk} - g'_{ijk}\tau^2 - h'_{ijk}\tau$$

$$a_{ijk} \equiv a'_{ijk} + r'_{ijk}\tau^2$$

$$b_{ijk} \equiv b'_{ijk} + m'_{ijk}\tau$$

$$e_{ijk} = e'_{ijk} + s'_{ijk}\tau^2$$

$$f_{ijk} = f'_{ijk} + n'_{ijk}\tau \tag{3.6}$$

$$\psi_{ijk} = c_{ijk} - a_{ijk}\xi^2 - b_{ijk}\xi - e_{ijk}\eta^2 - f_{ijk}\eta \tag{3.7}$$

On designating S_k for the kth τ region, we can write

$$S_k = \sum_{i\,j} S_{ijk}$$

$$S_{ijk} = \frac{k^2 \mathcal{R}^2 F^2 e^{-i\omega\tau}}{8\pi} \int_{\xi_{i-1}}^{\xi_i}\int_{\eta_{j-1}}^{\eta_j} \exp\left[- a_\xi \xi^2 - a_\eta \eta^2 + 2i\alpha\xi + 2i\beta\eta - 4\gamma^2\sigma^2(1 - \psi_{ijk})\right] d\xi\, d\eta \tag{3.8}$$

The integral has the form of the product of two complex error integrals. To use the tabulations, we transform the variables. Since the integrals on ξ and η have the same form, we use an integral on x and let it stand for either. Note that erf(z) and erfc(z) are defined by Abramowitz and Stegun(1965) as follows:

$$\mathrm{erf}(z) \equiv \frac{2}{\pi^{\frac{1}{2}}} \int_0^z e^{-t^2} dt$$

$$\operatorname{erfc}(z) = \frac{2}{\pi^{\frac{1}{2}}} \int_z^{\infty} e^{-t^2} dt$$

$$w(z) \equiv e^{-z^2} \operatorname{erfc}(-iz) \tag{3.9}$$

We now cast S_{ijk} into the form of (3.9) by defining the constant C_{ijk} and functions U_{ijk}, V_{ijk} as follows

$$S_{ijk} = \frac{k^2 \mathcal{R}^2 F^2}{32} C_{ijk} U_{ijk} V_{ijk} \tag{3.10}$$

$$C_{ijk} \equiv \exp\left[-4\gamma^2\sigma^2(1 - c_{ijk})\right] \tag{3.11}$$

$$U_{ijk} \equiv \frac{2}{\pi^{\frac{1}{2}}} \int_{\xi i-1}^{\xi i} \exp\left[-\frac{\xi^2}{A^2_{xijk}} - B_{xijk}\xi\right] d\xi \tag{3.12}$$

where with the aid of (2.23) and (3.7)

$$\frac{1}{A_x{}^2{}_{ijk}} \equiv a\xi + 4\gamma^2\sigma^2 a_{ijk}$$

$$B_{xijk} \equiv -i\alpha + 4\gamma^2\sigma^2 b_{ijk} \tag{3.13}$$

Similar expressions for the η dependence are

$$V_{ijk} \equiv \frac{2}{\pi^{\frac{1}{2}}} \int_{\eta_{j-1}}^{\eta} \exp\left[-\frac{\eta^2}{A_{yijk}} - B_{yijk}\eta\right] d\eta \tag{3.14}$$

$$\frac{1}{A^2_{yijk}} \equiv a_\eta + 4\gamma^2\sigma^2 e_{ijk}$$

$$B_{yijk} \equiv -i\beta + 4\gamma^2\sigma^2 f_{ijk} \tag{3.15}$$

Eq. (3.13) and (3.14) have the same form so we drop subscripts and cast it into the form of (3.9) by completing the square and changing variables as follows:

$$t = \frac{x}{A} + s$$

$$s = AB/2$$

$$g(i-1) = \frac{\xi_{i-1}}{A} + s \tag{3.16}$$

$$g(i) = \frac{\xi_i}{A} + s$$

$$U = Ae^{s^2} \int_{g(i-1)}^{g(i)} e^{-t^2} dt \tag{3.17}$$

$$U = A\, c^{s^2} \quad \mathrm{erfc}[g\,(i-1)] - \mathrm{erfc}\,[\,g(i)\,] \tag{3.18}$$

If g is real, (3.18) can be used for the evaluation of U and V. This would be expected for the specular direction when α and β are zero. For complex g, we change the form of erfc(g) to w(z) in (3.9).

Let $z = i\,g\,(i-1)$

then $$e^{s^2} \mathrm{erfc}\,[g\,(i-1)] = \exp\,[s^2 - g^2(i-1)]\;\; w[ig(i-1)] \tag{3.19}$$

Eq. 3.18 with the aid of (3.19) is

$$U = A\, e^{s^2} \left\{ e^{-g^2(i-1)}\, w[ig(i-1)] - e^{-g^2(i)} w[ig(i)] \right\} \tag{3.20}$$

V has the same form as U.

The substitution of A_{xijk}, B_{xijk}, etc. into (3.16) is simple and the evaluation (3.20) for U_{ijk} and V_{ijk} follows directly. I don't see much reason to write the expression because there is a chain of substitutions through to (3.5) for the τ dependence. The analytic evaluation of (3.8) is complete. From all of this mess, I would not expect the time dependence of the scattered signals to have a simple relationship to the time dependence of the surface.

IV. COHERENTLY SCATTERED SIGNAL

For my estimate of the coherently scattered signal, I will use the Eckart procedure and calculate the ensemble average of p. This method is very general and is applicable to a much wider range of conditions than is superfically apparent. By considering the more general problem, it is possible to use the dependence of the coherently reflected signal on γ to determine the probability density function of the surface W.

I begin by assuming the coherent signal is an ensemble average of many transmissions $\langle p \rangle$. The sound pressure is given by Helmholtz integral and the expansions of R and R', Eq. (2.11 and 2.12). For a moderately directive source and ζ very small relative to R and R', I can write

$$R + R' \simeq R(\zeta = 0) + R'(\zeta = 0) + 2\gamma\zeta \tag{4.1}$$

I use [space] to represent all of the non-random part of x, y, and t dependence of the Helmholtz integral,

$$p = \int_{-\infty}^{\infty} [\text{space}]\, e^{2i\gamma\zeta} ds \tag{4.2}$$

and

$$p_o = \int_{-\infty}^{\infty} [\text{space}]\, ds \quad \text{for } \zeta = 0 \tag{4.3}$$

The latter expression is exact for all configurations and (4.2) is only restricted by the approximation, (4.1). The ensemble average of (4.2) is

$$\langle p \rangle = \int_{-\infty}^{\infty} [\text{space}]\, ds \quad \langle e^{2i\gamma\zeta} \rangle \tag{4.4}$$

The average in (4.4) is given by

$$\left\langle e^{2i\gamma\zeta} \right\rangle = \int_{-\infty}^{\infty} W_1 e^{2i\gamma\zeta} d\zeta \tag{4.5}$$

and <u>this is the characteristic function of W_1</u>. Hence, we can write the following relations

$$\left[\langle p \rangle / p_o\right] = \int_{-\infty}^{\infty} W_1 e^{2i\gamma\zeta} d\zeta \tag{4.6}$$

$$W_1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \left[\langle p \rangle / p_o\right] e^{-2i\gamma\zeta} d\gamma \tag{4.7}$$

Since $\langle p \rangle / p_o$ is a function of positive $\gamma\sigma$, Eq. 4.7 should be regarded as the sine and cosine integrals from 0 to ∞. The normalized coherent reflection is the Fourier transformation of the pdf of the surface.

For a test, I use the experimental data of Mayo, Wright, and Medwin (70). They measured the surface distribution W_1 and $\left[\langle p \rangle / p_o\right]^2$ versus $4\gamma^2\sigma^2 = g$, where g is often referred to as the roughness parameter. If W_1 is Gaussian then

$$\left[\langle p \rangle / p_o\right]^2 = \exp(-g) \text{ for Gaussian } W_1 \tag{4.8}$$

however, their data did not fit (4.8). The data agreed quite well with (4.8) for g less than 4 (or $\gamma\sigma < 1$). At larger $\gamma\sigma$, the coherent component is orders of magnitude larger than predicted by (4.8). All aspects of the measurements were painstakingly re-examined, including finite illuminated area, but no "mistakes" were identified. The non-fit became the motivation of this research. For simplicity in a numerical test, I approximated their measured distribution function by the linear segments shown on Figure 4.1. The substitution and evaluation of (4.6) for the piece-wise linear function is routine and omitted here. A comparison of my calculation of $\left[\langle p \rangle / p_o\right]^2$ and their data are shown on Figure 4.2. The phase of the coherently reflected signal probably changes as $\gamma\sigma$ increases. The phase is needed in the evaluation of Eq. 4.7. Since the phase is needed for the inverse transform, I have not attempted to do it.

The procedure we have described, measure the reflected signal at $\sigma = 0$ and then the coherently reflected signal for the roughened surface is easy to do in the laboratory. Doing this for a rough sea surface would require highly accurate measurements and calculations. I suggest that an alternative procedure might be to measure the cross correlation of signals at a pair of separated hydrophones.

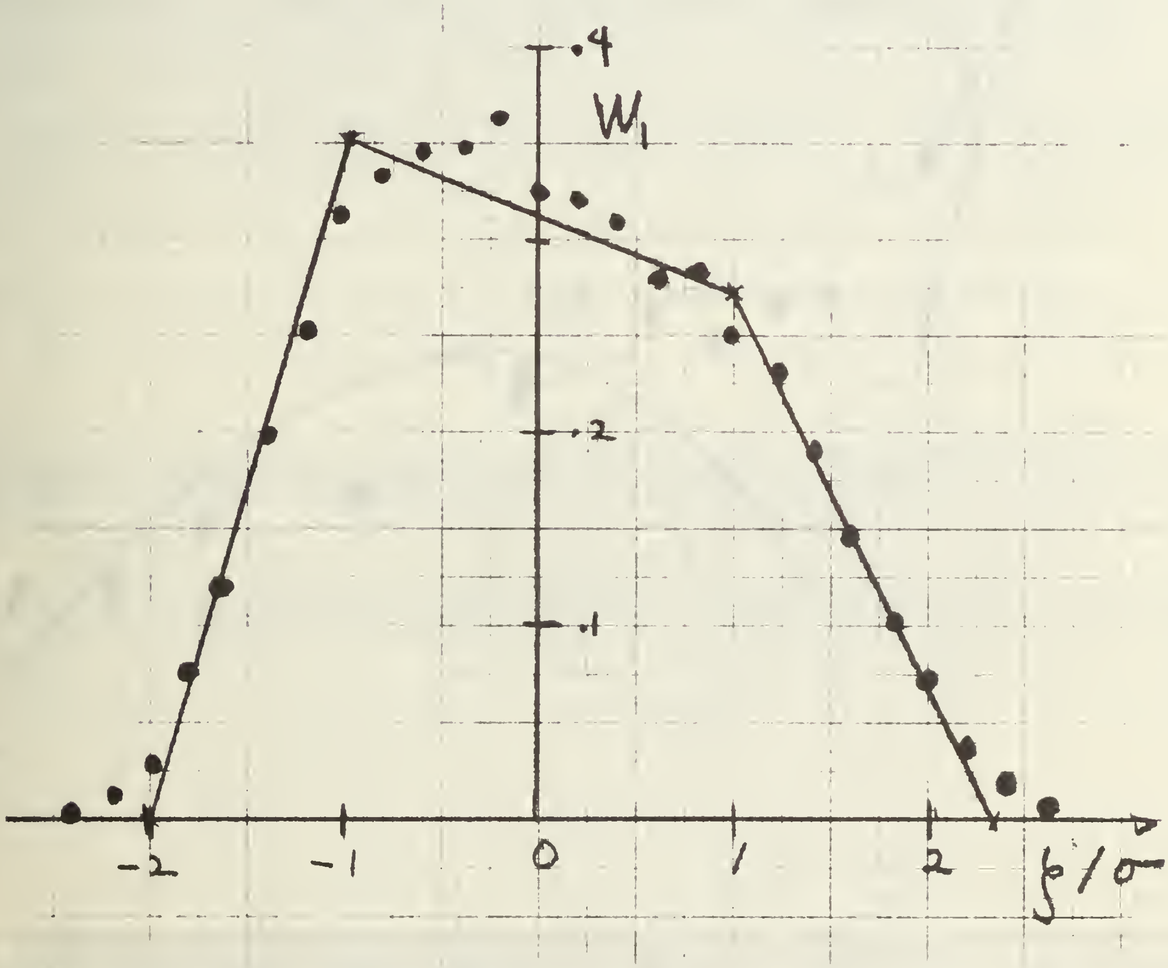

Fig. 4.1. Experimental probability density function of the model sea surface $\sigma = .45$ cm. The solid line is the piece-wise linear approximation to the distribution function. Data furnished by Mayo, Wright and Medwin.

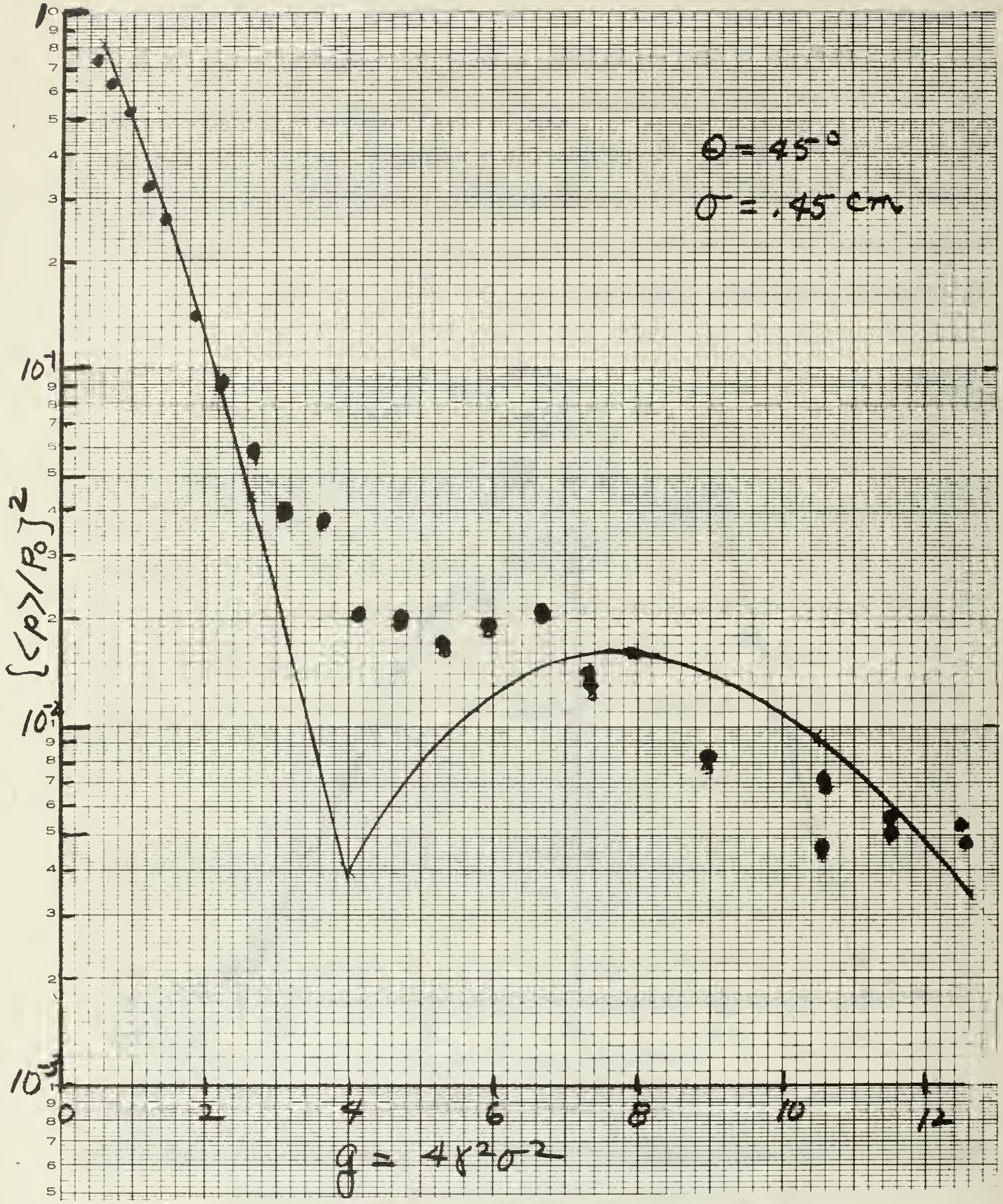

Fig. 4.2. Coherently reflected signal. The data points are from Mayo, Wright and Medwin (1970). The solid line is calculated for the probability density function shown on Figure 4.1. The data are for angle of incidence = 45° and σ = .45 cm.

V. COSINE CORRUGATED SURFACE

The surface of water waves often looks like a cosine corrugated surface for relatively long times and over fairly large areas. If one watches carefully, the phases and amplitudes of the wave change randomly. It is easy to make a cosine surface random analytically by letting the phase change randomly between each observation (or signal transmission). In addition, an ensemble can be formed of waves having different amplitudes. I assume the m^{th} wave surface is given

$$\zeta = \zeta_m \cos K (x - vt - x_m) \tag{5.1}$$

Where v is the wave velocity, ζ_m is the amplitude, and x_m is the phase. The procedure for calculating the scattered sound signal is to substitute (5.1) into (2.16) and to expand the result with the aid of the following expansion in Bessel functions

$$e^{ia \cos bx} = J_o(a) + 2\sum_1^\infty (i)^n J_n(a) \cos (nbx) \tag{5.2}$$

The result of integration and manipulation is

$$p = p_o e^{-\beta^2/a_y} \left\{ J_o(2\gamma\zeta_m) e^{-\alpha^2/a_x} + \sum_1^\infty (i)^n J_n(2\gamma\zeta_m) \exp\left[- \frac{(2\alpha + nK)^2}{4a_x} - inK(vt + x_m)\right] \right.$$
$$\left. + \sum_1^\infty (i)^n J_n(2\gamma\zeta_m) \exp\left[- \frac{(2\alpha - nK)^2}{4a_x} + inK(vt + x_m)\right] \right\} \tag{5.3}$$

where
$$p_o \equiv \frac{A_o \mathcal{R} B e^{-i\left[\omega t - k(R_1 + R_2)\right]}}{R_1 + R_2} \tag{5.4}$$

$$A_o \equiv \left[1 - i\left(1 - \frac{R_2}{R_x}\right)d_L\right]^{-1/2}\left[1 - i\left(1 - \frac{R_2}{R_y}\right)d_W\right]^{-1/2} \tag{5.5}$$

Before I wipe out most of the terms in (5.3) I should discuss them. The traveling water wave introduced an infinite series of time-dependent terms having frequencies nKv. These terms contribute a modulation of the reflected signal. The higher harmonics are more important at larger roughness, i.e. $\gamma\zeta_o$. If the reflected signal were processed by means of a spectrum analyzer, the spectrum would be like that shown on Figure 5.1. The powers in the components depend on the bistatic geometry.

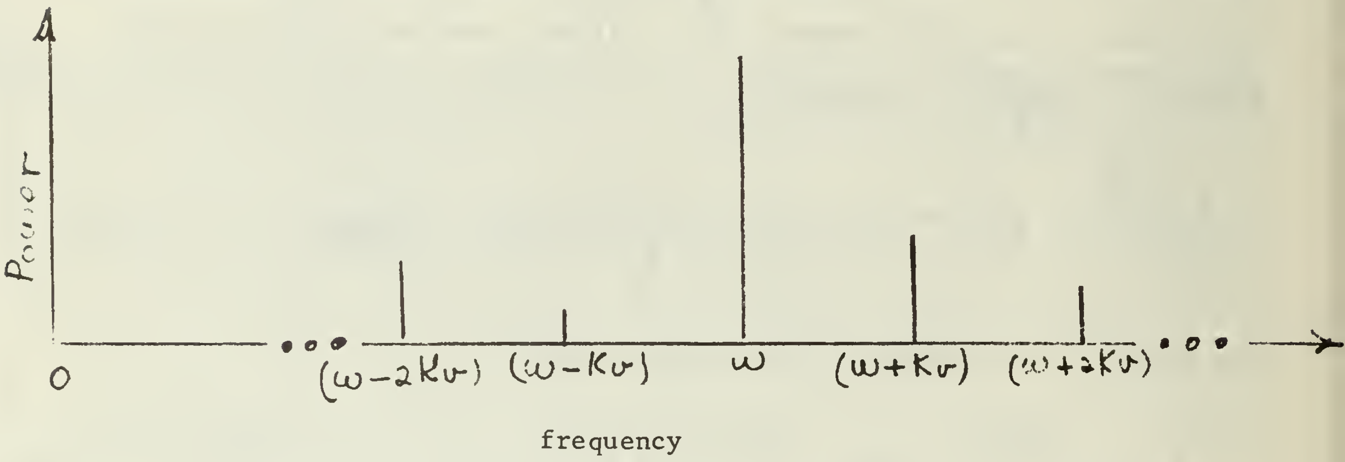

Fig. 5.1. Spectrum of the signal reflected at a traveling water wave. ω is the frequency of the sound signal and Kv is the frequency of the water wave.

The simplest ensemble average is obtained by letting all phases of the traveling wave be equally likely. An average over Kx_m from 0 to 2π eliminates all of the terms in the summation in Eq. (5.3). In the specular direction (5.3) then reduces to

$$\langle p \rangle / p_o = J_o(2\gamma\zeta_m) \tag{5.6}$$

The distribution function of a cosine wave is

$$\begin{aligned} \zeta &= \zeta_m \cos K x \\ W_c &= \pi^{-1}\left(\zeta_m^2 - \zeta^2\right)^{-1/2}, \quad \zeta < \zeta_m \\ W_c &= 0 \qquad \zeta > \zeta_m \end{aligned} \tag{5.7}$$

The circle can be completed by averaging (5.6) over a random set of amplitudes. For example, the envelope of a Gaussian random function has a Rayleigh distribution function:

$$W_R = \sigma^{-2}\, \zeta \exp\left[- \zeta^2/(2\sigma^2)\right] \tag{5.8}$$

$$\int_o^\infty J_o\,(2\gamma\zeta\,)\quad W_R d\zeta \;=\; \exp\left[-2\gamma^2\sigma^2\right] \tag{5.9}$$

This would be expected on the basis of the central limit theorem. I suggest that fair approximations to non-Gaussian functions can be made by averaging (5.6) for several values of ζ_m.

VI. TOTAL SIGNAL SCATTERED AT A NON-GAUSSIAN SURFACE

This is my last section and lest the reader has forgotten, my purpose is to do the inverse problem. The basic formulation of the scattering problem is given in Sections 2 and 3. There, the emphasis was on going from a known surface to calculated scattered signal. This analysis started with a bivarate Gaussian surface and its correlation function. I could have used the characteristic function and removed the Gaussian restriction as follows:

$$C_2 \equiv \left\langle e^{2i\gamma(\zeta-\zeta')} \right\rangle \tag{6.1}$$

The only change is to replace $\exp\left[-4\gamma^2\sigma^2(1-\psi)\right]$ by C_2 in Eq. (2.22). Following my procedure in Section 4, I evaluate p_o and express Eq. (2.22) as follows:

$$\langle pp^* \rangle / p_o^2 = (a_\xi\, a_\eta)^{\frac{1}{2}}\, \pi^{-1} e^{-i\omega\tau}$$

$$\iint D \exp\left[2i\alpha\xi + 2i\beta\eta\right] C_2 d_\xi d_\eta \tag{6.2}$$

$$D \equiv \exp\left[-\, a_\xi \xi^2 - a_\eta \eta^2\right] \tag{6.3}$$

Since a_ξ and a_η are functions of k, θ_1, θ_2, and θ_3, they can also be expressed as a function of α and β. I make a Fourier transform operation on assuming that D and D^{-1} are slowly varying functions of ξ and η. Integration of α and β yields $\delta(\xi - \xi')$ and $\delta(\eta - \eta')$ type functions and the expression for C_2 is the following

$$\frac{1}{2\pi^2} \iiint_{-\infty}^{\infty} (a_\xi\, a_\eta)^{-\frac{1}{2}} D^{-1} \frac{\langle pp^* \rangle}{p_o^2} e^{i\omega\tau} \exp\left[-i(\omega'\tau + 2\alpha\xi + 2\beta\eta)\right] d\alpha d\beta d\omega' = C_2 \tag{6.4}$$

The α and β dependences of a_ξ and a_η are greatly simplified by letting $\theta_1 = \pm\theta_2$, $\theta_3 = 0$, and holding R_1 and R_2 constant for a set of measurements. Obviously, one's ability to do the inverse transformation depends upon having accurate values of $\langle pp^* \rangle$. I assume the user would do the transforms numerically. Although the integral has infinite limits, actual measurements of $\langle pp^* \rangle$ give both upper and lower limits. There is another consideration. Since D operates as a spatial high pass filter in (6.2), I doubt if C_2 would have much accuracy at ξ and η much larger than ξ_f and η_f.

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Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Postgraduate School Monterey, California 93940	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

Coherent and Incoherent Components of Sound Scattered at a Time Dependent Rough Surface

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report, NPS-61Md72121A 31 December 1972

5. AUTHOR(S) (First name, middle initial, last name)

Clarence S. Clay

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
December 1972	39	13

8a. CONTRACT OR GRANT NO. b. PROJECT NO. c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Naval Postgraduate School Monterey, California 93940

13. ABSTRACT

Theoretical expressions are derived for the sound scattered at a time-dependent rough surface. The calculations are made for a Gaussian shaded source transducer and point receiver. The Helmholtz theorem and Fresnel approximation are used. The rough surface is assumed to be a traveling wave and to have a traveling wave packet type of correlation function. The coherent component of the signal is the product of the Fourier transformation of the surface distribution function and the smooth surface reflection signal. Comparison of theory and experiment shows the coherent component to be sensitive to the non-Gaussian character of the wind-blown water waves. The incoherent components and the temporal correlation function of the scattered sound are given. For the special case of a traveling cosine wave type of rough surface, spectrum of the scattered sound includes components which are multiples of the frequency of the surface wave. For surfaces describable by a bivariate Gaussian distribution function, the temporal correlation is a function of, but not the same as, the time dependence of the rough surface. The scattered sound is insensitive to the spatial correlation function of the surface at distances larger than the dimensions of the transducer divided by the cosine of the incident angle. The final expressions are complex error integrals and can be used for all values of roughness. This task was supported by Naval Ship Systems Command (Code PMS 388).

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Rough-surface transmission						
Sound scattering						
Coherent reflection						